Well-Quasi-Orderings on Binary Trees

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Abstract

We provide a set of "natural" requirements for quasi-orderings of finite and infinite binary trees. We show that any ordering satisfying these requirements must be well-ordered and that there is a unique minimal such ordering.

> *The checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops. To the pure mathematician it is natural to give an ordinal number. In this problem the ordinal might be* $(n-r)\omega^2 + (r-s)\omega + k$. *A less highbrow form of the same thing would be to give the integer* $2^{80}(n-r) + 2^{40}(r-s) + k$.
>
> —Alan M. Turing (1949)

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1 Introduction

Introducing a completely new structure to a problem can at times lead to an elegant solution. The structure of a well-ordered set has the important properties of being total (linear) and having every strictly descending sequence be finite.

Tree structures are a common data representation in software and can serve as the state of an algorithm. A well-ordering on tree structures can be used to make conclusions about an algorithm whose state includes a tree structure. For instance, an algorithm whose changes to a tree structure are always strictly descending in a well-ordering of trees must be an algorithm that terminates.

An easy way to define an ordering is to enumerate a set of requirements, or principles, that the ordering must satisfy and then choose the minimum ordering which satisfies those principles. In this paper we introduce a set of principles for orderings of binary trees (both finite and infinite) which result in well-orderings and a minimal quasi-ordering of binary trees satisfying those principles.

2 Basics

The *binary tree* is the basic object in this paper. Since this paper deals only with trees that are binary, any reference to *tree* refers to the binary variety. A tree consists of nodes which in their most distilled form are nothing more than positions in a tree. This position is determined by the path of "left" and "right" segments from the root of a tree. It is natural to define such paths as strings over a two letter alphabet, a letter for right, say "0", and a letter for left, say "1". Formally, a node will only be a path. A tree is thus a set of node-paths, in other words, a language. Since all non-root nodes of a tree have a parent, all prefixes of a node-path must also be a node of a tree. Thus trees, as a language, are closed over truncation. This lead to a formal representation of binary trees as formal languages over the alphabet $\{\mathbf{0}, \mathbf{1}\}$ which are closed under truncation.

Definition 1. Let $\mathbb{T}$, all binary trees, be the collection of all languages x over the alphabet $\{\mathbf{0}, \mathbf{1}\}$ such that for any string s in x, any prefix of s is also in x.

The root node of a tree is the empty string ε. For example, the tree with a root node and only a left leaf node is $\{\varepsilon, \mathbf{1}\}$. The empty tree, $\emptyset$, is $\{\}$.

With a formal definition of a binary tree we proceed to define a fundamental operation: extracting a subtree.

Given a node in a tree, the subtree at that node consists of all the nodes with their relative structure below that node. A node is simply a string path so the subtree at that node consists of all the string paths from the given node to lower nodes in the subtree. We formalize this operation as $x|_s$ where x is a tree and s a string (node path):

Definition 2. For any tree x and string $s \in \{\mathbf{0}, \mathbf{1}\}^*$,

$$x|_s \quad := \quad \{r : sr \in x\}$$

Definition 3. Let $\mathbb{T}_f$ be the collection of all finite trees in $\mathbb{T}$.

With trees as formal languages, it is worth reviewing some concatenation operations.

Definition 4. For any language x and string $w \in \{\mathbf{0}, \mathbf{1}\}^*$,

$$\begin{aligned} xw &:= \{sw : s \in x\} \\ wx &:= \{ws : s \in x\} \end{aligned}$$

For any languages x_1 and x_2,

$$x_1 x_2 \quad := \quad \{s_1 s_2 : s_1 \in x_1, s_2 \in x_2\}$$

The following operation will prove very convenient in constructing trees.

Definition 5. For any trees x and y,

$$(x \curlywedge y) \quad := \quad \mathbf{1}x \cup \mathbf{0}y \cup \{\epsilon\}$$

Since the letters $\mathbf{1}$ and $\mathbf{0}$ are used to indicate “left” and “right” respectively, negation is used to “switch” from side to side, thus $\overline{\mathbf{0}} = \mathbf{1}$ and $\overline{\mathbf{1}} = \mathbf{0}$.

2.1 Ordering Trees

The anti-symmetric property of partial orderings proves to be inconvenient in being too constrained. More convenient are the binary relationships called quasi-orderings which are reflexive and transitive, but not necessarily anti-symmetric. When it is necessary to work in a strict partial ordering, any quasi-ordering of trees determines a partial ordering over equivalence classes of trees.

Given a quasi-ordering A, $(x,y) \in A$ is denoted $x \leq_A y$ and $y \geq_A x$. Strict inequality $x <_A y$ is defined as "$x \leq_A y$ and not $y \leq_A x$". If $x \leq_A y$ and $y \leq_A x$ then $x \simeq_A y$ and thus x and y are in the same equivalence class of the partial ordering determined by A.

Quasi-orderings can also have the quality of being total (all elements can be compared) and having no strictly descending sequence (well-quasi-orderings).

The first class of tree orderings to consider satisfies two intuitive principles, Growth and Monotonicity.

The Growth Principle states that a tree is larger that its subtrees.

Principle 1 (Growth). An ordering A satisfies the Growth principle over $\mathcal{C}$ when for any trees x_1 and x_0 in $\mathcal{C}$,

$$\begin{array}{rcl} x_1 & \leq_A & (x_1 \curlywedge x_0) \\ x_0 & \leq_A & (x_1 \curlywedge x_0) \end{array}$$

The other intuitive principle is that taking a tree and then replacing a branch with a larger branch results in a larger tree.

Principle 2 (Monotonicity). An ordering A satisfies the Monotonicity principle over $\mathcal{C}$ when for any trees x_1, x_2 and y in $\mathcal{C}$,

$$x_1 \leq_A x_2 \rightarrow \left\{ \begin{array}{l} (x_1 \curlywedge y) \leq_A (x_2 \curlywedge y) \\ (y \curlywedge x_1) \leq_A (y \curlywedge x_2) \end{array} \right.$$

Definition 6. For any collection of trees $\mathcal{C}$, let $\mathbf{\Omega_1}(\mathcal{C})$ be the collection of all quasi-orderings on $\mathcal{C}$ which satisfy Growth and Monotonicity over $\mathcal{C}$.

One of the nice properties of $\mathbf{\Omega_1}(\mathcal{C})$ is that is is closed under arbitrary intersection. Thus $\bigcap \mathbf{\Omega_1}(\mathcal{C})$ is in $\mathbf{\Omega_1}(\mathcal{C})$.

It turns out that $\bigcap \mathbf{\Omega_1}(\mathbb{T}_f)$ is the relationship of homeomorphic embedding on finite trees:

Definition 7. For any finite trees x and y, x is homeomorphically embedded in y if and only if

$$\begin{array}{ll} x = y & \text{OR} \\ x \text{ is homeomorphically embedded in } y|_{\mathbf{1}} & \text{OR} \\ x \text{ is homeomorphically embedded in } y|_{\mathbf{0}} & \text{OR} \end{array}$$

$x|_{\mathbf{0}}$ is homeomorphically embedded in $y|_{\mathbf{0}}$ and $x|_{\mathbf{1}}$ is homeomorphically embedded in $y|_{\mathbf{1}}$

As a special case of Higman's Lemma , we know that, in any infinite sequence $\{t_i\}_{i<\omega}$ of finite binary trees, there must be two trees t_j and t_k ($j < k$) such that t_j is embedded in t_k. In other words an infinite strictly descending sequence of trees is impossible in $\bigcap \mathbf{\Omega_1}(\mathbb{T}_f)$ and thus all orderings in $\mathbf{\Omega_1}(\mathbb{T}_f)$.

Thus we find that all orderings in $\mathbf{\Omega_1}(\mathbb{T}_f)$ are well-quasi-ordered. Orderings in $\mathbf{\Omega_1}(\mathbb{T}_f)$ however are not necessarily well-ordered since they may not be total. For instance, what should the ordering of $(x \curlywedge y)$ and $(x' \curlywedge y')$ be when $x' \leq x$ but $y \leq y'$? There is no preference given to "left" or "right".

To narrow down a smaller ordering collection we will make "left" more significant than "right" by borrowing the idea of a lexicographic (dictionary) ordering. A lexicographic ordering on two letter acronyms would state that $l_0 l_1 \leq l_2 l_3$ when $l_0 < l_2$ regardless of how l_1 and l_3 relate.

To extend this idea to binary trees we might state that tree x is less than tree y whenever x's left branch is strictly less than y's left branch regardless of how x and y's right branches relate. The only problem with this attempt is that with orderings in $\mathbf{\Omega_1}(\mathcal{C})$ x's right branch could be vastly greater than all of y in which case making x less than y wouldn't make much sense. For this reason we will modify our attempt at a lexicographic rule for binary trees by adding the condition that x's right branch must be less than y before can declare that x is less than y. With this addition we having the following formal principle.

Principle 3 (Lexicography). An ordering A satisfies the Lexicography principle over $\mathcal{C}$ when for all trees x_1,x_0,y_1, and y_0 in $\mathcal{C}$,

$$x_1 <_A y_1 \text{ and } x_0 \leq_A (y_1 \curlywedge y_0) \rightarrow (x_1 \curlywedge x_0) \leq_A (y_1 \curlywedge y_0)$$

With this added principle we let $\mathbf{\Omega_{1L}}(\mathcal{C})$ be the collection of all orderings on $\mathcal{C}$ which satisfy Growth, Monotonicity and Lexicography over $\mathcal{C}$.

The first new interesting property of all the orderings in $\mathbf{\Omega_{1L}}(\mathcal{C})$ is that they are not all strict partial ordering.

In general,

Lemma 1. *[Lexicographic Collapse] For any ordering A in $\mathbf{\Omega_{1L}}(\mathcal{C})$, if $x <_A y$ then*

$$(x \curlywedge (y \curlywedge z)) \quad \simeq_A \quad (y \curlywedge z)$$

Proof.

$$\begin{array}{lll} (x \curlywedge (y \curlywedge z)) & \leq_A \quad (y \curlywedge z) & \text{by Lexicography} \\ (x \curlywedge (y \curlywedge z)) & \simeq_A \quad (y \curlywedge z) & \text{by Growth} \end{array}$$

□

2.2 Well-Orderings on Finite Trees

Restricting ourselves to finite trees, we find that all orderings in $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$ are total.

Theorem 1. *For any ordering A in $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$,*

$$x \leq_A y \text{ or } y \leq_A x$$

for all finite trees x and y.

Proof. Let A be any ordering in $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$. We proceed with induction on n with the following inductive proposition :

> For all finite trees x and y where $n \geq |x| + |y|$,
> $$x \leq_A y \quad \text{or} \quad y \leq_A x$$

For $n = 0$, x and y could only be $\emptyset$, thus case $n = 0$ holds.

Assume case $n = N$ holds. Consider any finite trees x and y with $|x| + |y| = N + 1$. Since $|x|_1| + |y|_1| \leq N$ we must have three cases relating $x|_1$ and $y|_1$:

Case $y|_1 \simeq_A x|_1$. Either $x|_0 \leq_A y|_0$ or $y|_0 \leq_A x|_0$ and thus either $x \leq_A y$ or $y \leq_A x$.

Case $y|_1 <_A x|_1$: Since $|x| + |y|_0| \leq N$ either $y|_0 \leq_A x$ in which case $y \leq_A x$ from Lexicography or $x \leq_A y|_0$ in which case $x \leq_A y$ from Growth.

Case $y|_1 >_A x|_1$: Likewise. □

With $\mathbf{\Omega_1}(\mathbb{T}_f)$ we found well-quasi-ordering that are not necessarily total. Introducing the Lexicography principle we narrowed this ordering collection down to $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$ which does consists of total quasi-orderings and thus well-orderings. This leads us to ask what well-orderings can be found in $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$. Okada and Steele [8] relate any ordering in $\mathbf{\Omega_1}(\mathbb{T}_f)$ to Ackermann's ordinal notation. Upon inspection of the recursive ordering relationships of $\omega^\alpha + \beta$ ordinals in ϵ_0 we find a striking similarity to the Lexicography principle and thus we can hypothesize that the following mapping from trees to ordinals might preserve an ordinal ordering to an ordering in $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$.

We recursively define a mapping, m_1, from finite trees into ϵ_0 (the set of all ordinals up to ϵ_0).

Definition 8.

$$\begin{aligned} m_1(\emptyset) &:= 0 \\ m_1((x \curlywedge y)) &:= \omega^{m_1(x)} + m_1(y) \end{aligned}$$

Definition 9. Let M_1 be the quasi-ordering on finite trees

$$\{(x, y) : m_1(x) \leq m_1(y)\}$$

Indeed M_1 must be a quasi-ordering since ϵ_0 (with the usual ordering) is also reflexive and transitive.

Theorem 2. *The quasi-ordering M_1 is in $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$.*

Proof. It suffices to show M_1 satisfies Principles Growth, Monotonicity and Lexicography.

For any finite trees x_1 and x_0,

$$\left.\begin{matrix} m_1(x_1) \\ m_1(x_0) \end{matrix}\right\} \leq \omega^{m_1(x_1)} + m_1(x_0)$$

Thus by definition of m_1 and M_1, for any finite trees x_1 and x_0,

$$\left.\begin{matrix} x_1 \\ x_0 \end{matrix}\right\} \leq_{M_1} (x_1 \curlywedge x_0)$$

Thus M_1 satisfies the Growth principle.

Consider any finite trees x_1, x_0, and y with $x_0 \leq_{M_1} x_1$. Immediately $m_1(x_0) \leq m_1(x_1)$ and thus $\omega^{m_1(x_0)} + m_1(y) \leq \omega^{m_1(x_1)} + m_1(y)$ and $\omega^{m_1(y)} + m_1(x_0) \leq \omega^{m_1(y)} + m_1(x_1)$. By definition of m_1 and M_1, $(x_0 \curlywedge y) \leq_{M_1} (x_1 \curlywedge y)$ and $(y \curlywedge x_0) \leq_{M_1} (y \curlywedge x_1)$. Thus M_1 satisfies the Monotonicity principle.

Consider any finite trees x_1, x_0, y_1, and y_0,

$$y_1 <_{m_1} x_1 \text{ and } y_0 \leq_{m_1} (x_1 \curlywedge x_0)$$

$$m_1(y_1) < m_1(x_1) \text{ and } m_1(y_0) \leq \omega^{m_1(x_1)} + m_1(x_0)$$

$$\omega^{m_1(y_1)} + m_1(y_0) \leq \omega^{m_1(x_1)} + m_1(x_0)$$

$$(y_1 \curlywedge y_0) \leq_{M_1} (x_1 \curlywedge x_0)$$

Thus M_1 satisfies the Lexicography principle. ☐

Since m_1 maps trees to ordinals, we can only be sure M_1 has order type ϵ_0 once we know m_1 maps onto all of ϵ_0.

Theorem 3. *M_1 is of order-type ϵ_0*

Proof. Since every ordinal in ϵ_0 can be expressed as some combination of $\omega^\alpha + \beta$ and 0, m_1 maps onto ϵ. Thus, as a quasi-ordering, M_1 is of order-type ϵ_0. ☐

Since M_1 is order-isomorphic to ϵ_0 and ϵ_0 induction is equivalent to the consistency of Peano Arithmetic, this means that the Embedding Lemma of Higman cannot be proved in Peano Arithmetic [2].

Now that we have a collection of well-orderings and a member of it, M_1, it is natural to wonder whether M_1 is the minimum ordering of $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$. If orderings in $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$ are viewed as sets of comparisons and compared via set-wise inclusion, then there is no minimum orderings of $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$. Proving this surprising result is a modification of a proof in [1]. However, M_1 is the *leanest* ordering of $\mathbf{\Omega_{1L}}(\mathbb{T}_f)$ as proved as an example in [1].

2.3 Arithmetic

The mapping from ordinals to finite binary trees gives a convenient data structure for representing ordinals below ϵ_0. Arithmetic operations (commutative addition $\oplus$, commutative multiplication $\otimes$, and exponentiation), and a predecessor operation to get fundamental sequences, are now easy to define; the following correspondences are suggestive:

$$\begin{array}{rcl}
0 & \mapsto & \emptyset \\
1 & \mapsto & (\emptyset \curlywedge \emptyset) \\
x \oplus \emptyset & \mapsto & x \\
x \otimes \emptyset & \mapsto & \emptyset \\
(x \curlywedge \emptyset) \otimes (x' \curlywedge y') & \mapsto & (x \oplus x' \curlywedge (x \curlywedge \emptyset) \otimes y') \\
(x \curlywedge y) \otimes z & \mapsto & ((x \curlywedge \emptyset) \otimes z) \oplus (y \otimes z) \\
\omega^x & \mapsto & (x \curlywedge \emptyset)
\end{array}$$

$$\begin{array}{rcll}
(x \curlywedge y) \oplus (x' \curlywedge y') & \mapsto & (x \curlywedge y \oplus (x' \curlywedge y')) & x \leq x' \\
\mathit{pred}_n((\emptyset \curlywedge \emptyset)) & \mapsto & \emptyset & \\
\mathit{pred}_n((x \curlywedge \emptyset)) & \mapsto & (\mathit{pred}_n(x) \curlywedge \emptyset) \otimes n & x \in \mathrm{Succ} \\
\mathit{pred}_n((x \curlywedge \emptyset)) & \mapsto & (\mathit{pred}_n(x) \curlywedge \emptyset) & x \in \mathrm{Lim} \\
\mathit{pred}_n((x \curlywedge y)) & \mapsto & (x \curlywedge \mathit{pred}_n(y)) & y \not\equiv \emptyset
\end{array}$$

For example, this binary-tree data structure could be used in implementing the computation of the various extensions of Ackermann's function (see, for example, [4]). An ordinal-indexed function $A_\alpha(n)$ can be defined for ordinals α and natural numbers n by

$$A_\alpha(n) = \begin{cases} 2n & \text{if } \alpha = 0, n \geq 1, \\ A_\beta^{(n)}(1) & \text{if } \alpha \text{ is a successor ordinal } \beta + 1, \\ A_{pred_n(\alpha)}(n) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

The computation of this function plays an important role in the unbounded search procedures of Reingold and Shen [9] . Moreover, these search procedures themselves use ordinals to index the recursive calls.

These operations also make it easy to encode problems like the "Battle of Hydra and Hercules" of Kirby and Paris [5] as hard-to-prove-well-defined functions on binary trees.

3 Infinite Trees

Working with infinite trees it will be useful to replace deeply nested branches of a tree. To replace the branch of tree x at node w with the tree y we use the following definition of *replacement*.

Definition 10. For any trees x and y and node $w \in x$,

$$x\left[y\right]_w := wy \cup \{s \in x : w \text{ not a prefix of } s\}$$

Lemma 2 (Repeated Monotonicity). *For any ordering A in $\mathbf{\Omega_1}(\mathbb{T})$, trees x, y and string $s \in x$,*

$$y \leq_A x|_s \Rightarrow x\left[y\right]_s \leq_A x$$

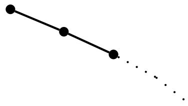
Proof. Use the Monotonicity principle recursively on x for each letter in s. □

When working with infinite trees it will be useful to denote the "truncation closure".

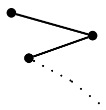
Definition 11. For any set of strings S let S^- denote the closure of S under string truncation, that is, the set of all prefixes of member strings of S.

For example, $\{\mathbf{000}\}^- = \{\varepsilon, \mathbf{0}, \mathbf{00}, \mathbf{000}\}$. Note that a prefix of a string can be itself.

Consider the infinite tree $\{\mathbf{0}\}^*$:



and $(\{\mathbf{01}\}^*)^-$:



both of which are not necessarily comparable in $\mathbf{\Omega_1}(\mathbb{T})$.

To see this consider the quasi-ordering of the following five equivalence classes

$$\begin{array}{lcl} P_1 & = & \{\emptyset\} \\ P_2 & = & \{\text{ all non-empty finite trees }\} \\ P_3 & = & \{\{\mathbf{0}\}^*\} \\ P_4 & = & \{(\{\mathbf{01}\}^*)^-, (\{\mathbf{10}\}^*)^-\} \\ P_5 & = & \{\text{ all other trees }\} \end{array}$$

ordered such that

- $\emptyset$ is strictly less than all non-empty trees
- all non-empty finite trees are strictly less than all infinite trees
- $\{\mathbf{0}\}^*$ is strictly less than all trees in P_5
- both $(\{\mathbf{01}\}^*)^-$ and $(\{\mathbf{10}\}^*)^-$ are strictly less than all trees in P_5
- $\{\mathbf{0}\}^*$ remains incomparable to both $(\{\mathbf{01}\}^*)^-$ and $(\{\mathbf{10}\}^*)^-$

This quasi-ordering satisfies Growth, Monotonicity and Lexicography yet $\{\mathbf{0}\}^*$ and $(\{\mathbf{01}\}^*)^-$ remain incomparable.

It seems natural that $\{\mathbf{0}\}^*$ should be less than $(\{\mathbf{01}\}^*)^-$ for the same reason $\{\mathbf{00}\}^-$ is less than $\{\mathbf{0101}\}^-$ by iterating Growth and Monotonicity.

However, for every N we have $\mathbf{0}^N\{\mathbf{01}\}^{*-}$ less than $(\{\mathbf{01}\}^*)^-$. It seems reasonable that $\{\mathbf{0}\}^*$ should also be less than $(\{\mathbf{01}\}^*)^-$.

$\{\mathbf{0}\}^*$ is the *limit* of the *sequence* of trees $\left\{\mathbf{0}^N\{\mathbf{01}\}^{*-}\right\}_N$.

To define a limit of trees formally we rely on the following definitions:

Definition 12. For any sequence of trees $\{x_i\}_i$,

$$\begin{array}{lcl} \liminf\limits_{i\to\infty} x_i & := & \bigcup\limits_{N=0}^{\infty} \bigcap\limits_{i=N}^{\infty} x_i \\ \limsup\limits_{i\to\infty} x_i & := & \bigcap\limits_{N=0}^{\infty} \bigcup\limits_{i=N}^{\infty} x_i \end{array}$$

When both the lim sup and the lim inf equal each other then the limit exists:

$$\lim_{i\to\infty} x_i = \liminf_{i\to\infty} x_i = \limsup_{i\to\infty} x_i$$

The construction of $\mathbf{0}^N\{\mathbf{01}\}^{*^{-}}$ via repeated application of tree replacement is a technique worth formalizing as a *tree series.*

Definition 13. Given a tree y and sequences of strings s_i and of trees x_i, a tree series is the result of repeated application of tree replacement, denoted

$$y \mathop{\bigtriangleup}_{i=0}^{n} [x_i]_{s_i} \quad = \quad y\,[x_0]_{s_0}\,[x_1]_{s_1} \ldots [x_n]_{s_n}$$

An infinite tree series is

$$y \mathop{\bigtriangleup}_{i=0}^{\infty} [x_i]_{s_i} \quad = \quad \lim_{n\to\infty} y \mathop{\bigtriangleup}_{i=0}^{n} [x_i]_{s_i}$$

We also definte three important properties of an infinite tree series.

Definition 14. Given an infinite tree series

$$y \mathop{\bigtriangleup}_{i=0}^{\infty} [x_i]_{s_i}$$

it is *extended* iff

$$\lim_{i\to\infty} |s_i| \quad = \quad \infty$$

and descreasing iff for every $n \geq 0$

$$x_n \quad \leq_A \quad \left.\left(x_0 \mathop{\bigtriangleup}_{i=0}^{n-1} [x_i]_{s_i} \right)\right|_{s_n}$$

Note that decreasing does NOT imply strict decreasing. A series can consist entirely of order-equivalent trees.

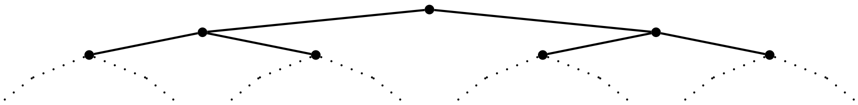
With these definitions in place, we can now introduce the Continuity principle.

Principle 4 (Continuity). An ordering A satisfies the Continuity principle iff

$$y \mathop{\bigtriangleup}_{i=0}^{\infty} [x_i]_{s_i} \quad \leq_A \quad y$$

for every decreasing extended tree series.

Surprisingly the addition of the Continuity princple does not force the tree $\{\mathbf{0},\mathbf{1}\}^*$



to be larger than $\emptyset$. The following quasi-ordering A satisfies Growth, Monotonicity, Lexicography and Continuity,

$$\begin{array}{rl} \emptyset & <_A \\ & \\ \{\mathbf{0},\mathbf{1}\}^* & <_A \end{array} \left\{ \begin{array}{c} \text{all} \\ \text{other} \\ \text{trees} \end{array} \right\}$$

but $\emptyset$ and $\{\mathbf{0},\mathbf{1}\}^*$ remain incomparable.

To address this we introduce the Nil principle,

Principle 5 (Nil). An ordering A satisfies the Nil principle over $\mathcal{C}$ when for any tree x in $\mathcal{C}$,

$$\emptyset \leq_A x$$

Now we define $\mathbf{\Omega_2}(\mathcal{C})$ to be the set of all quasi-orderings of $\mathcal{C}$ which satisfy the Growth, Monotonicity, Continuity and Nil principles over $\mathcal{C}$. Note $\mathbf{\Omega_2}(\mathbb{T}_f) = \mathbf{\Omega_1}(\mathbb{T}_f)$. Also note that the intersection of all quasi-orderings in $\mathbf{\Omega_2}(\mathbb{T})$ must be a member of $\mathbf{\Omega_2}(\mathbb{T})$.

Define $\mathbf{\Omega_{2L}}(\mathcal{C})$ to be the set of all quasi-orderings of $\mathcal{C}$ in $\mathbf{\Omega_2}(\mathcal{C})$ and satisfying Lexicography.

3.1 Homeomorphic Embedding

The homeomorphic embedding relationship for finite trees is equal to $\bigcap \mathbf{\Omega_1}(\mathbb{T}_f)$. Is this true in the infinite case? Is $\bigcap \mathbf{\Omega_2}(\mathbb{T})$ equal to the homeomorphic embedding over infinite trees? First we need to define the homeomorphic embedding over infinite trees.

Definition 15. A mapping h from a tree x to a tree y is a *homeomorphic embedding* iff h is one-to-one and $h(w)a$ is a prefix of $h(wa)$ for every $a \in \{\mathbf{0},\mathbf{1}\}$ and string wa in x.
If $y = \{h(w) : w \in x\}^-$ then h *covers* y.

Theorem 4. *The inverse of a homeomorphic embedding preserves the string prefix relationship* $\preceq$.

Proof. Suppose h is a homeomorphic embedding. If u is not a prefix of v then there exists a common prefix w with wa a prefix of u and $w\overline{a}$ a prefix of v. Furthermore, we must have both $h(w)a$ and $h(w)\overline{a}$ a prefix of $h(u)$ and $h(v)$, respectively, implying that $h(u)$ is not a prefix of $h(v)$. ☐

We can now show that, like in the finite case, $\bigcap \mathbf{\Omega_2}(\mathbb{T})$ is the homeomorphic embedding for infinite trees.

Theorem 5. *The homeomorphic embedding relationship is a subset of any quasi-ordering in* $\mathbf{\Omega_2}(\mathbb{T})$

Proof. Let A be any member of $\mathbf{\Omega_2}(\mathbb{T})$. Consider any homeomorphic embedding h from tree x into tree y. Let $\{s_i\}_{i=0}^{\infty}$ be any sequence enumerating all strings in $x\,\{\mathbf{0},\mathbf{1}\}$ with shorter strings always before longer strings. For all $n \geq 0$, define

$$\begin{aligned} z_n &:= \begin{cases} y|_{h(s_n)} & \text{if } s_n \in x \\ \emptyset & \text{otherwise} \end{cases} \\ t_n &:= \left. \left(y \mathop{\Delta}_{i=0}^{n-1} [z_i]_{s_i} \right) \right|_{s_n} \end{aligned}$$

Since

$$x \;=\; y \mathop{\Delta}_{i=0}^{\infty} [z_i]_{s_i}$$

showing that $z_n \leq_A t_n$ for all $n \geq 0$ shows that by Continuity that $x \leq y$ which completes the proof.

We finish with three cases for s_n.

1. For $s_n \notin x$ we have $z_n = \emptyset \leq_A t_n$.

2. For $s_0 \in x$ we have $s_n = \varepsilon$ thus $z_0 = y|_{h(\varepsilon)} \leq_A y = t_0$.

3. For $s_n \in x$ with $n > 0$ we have $s_n = s_k a$ for some $k < n$ and $a \in \{\mathbf{0},\mathbf{1}\}$. Since h is a homeomorphic embedding we have

$$y|_{h(s_k a)} \;\leq_A\; y|_{h(s_k)a}$$

With

$$\begin{aligned} z_n &= y|_{h(s_k a)} \\ y|_{h(s_k)a} &= z_k|_a \\ &= \left(y \bigtriangleup_{i=0}^{n-1} [z_i]_{s_i} \right)\Bigg|_{s_k a} \\ &= t_n \end{aligned}$$

we conclude $z_n \leq_A t_n$.

□

Corollary 1. *For any ordering $A \in \mathbf{\Omega_2}(\mathbb{T})$ and trees x and y,*

$$x \subseteq y \Rightarrow x \leq y$$

From these results we find a group of infinite trees which are always equivalent to each other in any ordering of $\mathbf{\Omega_2}(\mathbb{T})$.

Definition 16. Let $\mathbb{T}_h$ be the set of all trees x with $\{\mathbf{0},\mathbf{1}\}^*$ homeomorphically embedded in x.

Corollary 2. *For any A in $\mathbf{\Omega_2}(\mathbb{T})$,*

$$x \simeq_A y$$

for all trees x and y in $\mathbb{T}_h$.

Proof. For all trees x in $\mathbb{T}_h$, $x \subseteq \{\mathbf{0},\mathbf{1}\}^*$ and $\{\mathbf{0},\mathbf{1}\}^*$ is homeomorphically embedded in x. □

Theorem 6. *The homeomorphic embedding relationship is a member of $\mathbf{\Omega_2}(\mathbb{T})$*

Proof. The homeomorphic embedding must satisfy the Growth principle because for every infinite tree x and letter $a \in \{\mathbf{0},\mathbf{1}\}$ we have $x|_a$ homeomorphically embedded in x with $h(s) = as$.

The homeomorphic embedding must also satisfy the Monotonicity principle since for any tree x, letter $a \in \{\mathbf{0},\mathbf{1}\}$, and homeomorphically embedding h_1 from tree z into $x|_a$ we have $x\,[z]_a$ homeomorphically embedded in x by

$$\begin{aligned} h_2(as) &= ah_1(s) \\ h_2(\overline{a}s) &= \overline{a}s \end{aligned}$$

Showing that the homeomorphic embedding satisfies the Continuity principle is a little bit more involved. Consider any extended decreasing tree series

$$z_n \;:=\; y \bigtriangleup_{i=0}^{n-1} [x_i]_{s_i}$$

By extending the logic for Monotonicity we conclude there exists a sequence of homeomorphic embeddings h_n which map z_n into y. Define

$$t_n \;:=\; \{w \in z_n : \text{for all } j \geq n,\ s_j \text{ not a prefix of } w\}$$

Since

$$\bigcup_{n=0}^{\infty} t_n \;=\; \lim_{n\to\infty} z_n$$

we have the well-defined homeomorphic embedding

$$h(w) \;:=\; h_{\min\{i:\, w \in t_i\}}(w)$$

from $\lim_{n\to\infty} z_n$ to y. ☐

With Theorems 5 and 6 we can conclude that the homeomorphic embedding relationship equals $\cap\, \mathbf{\Omega_2}(\mathbb{T})$.

Using Higman's Lemma [3] and results from [6] it can be shown that the homeomorphic embedding as defined above is well-quasi-ordered. A binary tree in this paper corresponds to a "Q-tree" in [6] with Q as the mapping from strings in a tree to their respective sub-branch. Sub-branches are ordered by another Q-tree ordering with Q as the identity mapping and strings ordered with the quasi-ordering described in Higman's Lemma [3].

In this paper however we will explicitly prove that any ordering in $\mathbf{\Omega_2}(\mathbb{T})$ is well-quasi-ordered rather than relying on the homeomorphic relationship being well-quasi-ordered.

3.2 Pruning

Before analyzing the quasi-orderings in $\mathbf{\Omega_{2L}}(\mathbb{T})$, we decompose some of the structure within infinite trees. The first notable component of an infinite tree is the presence of infinite sequences of strings.

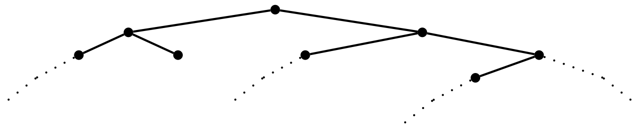
Definition 17. A tree is *linear* iff for any two member strings one must be a prefix of the other.

Every infinite trees has a fundamental structure of all the infinite linear trees contained by it. The union of all such infinite linear subtrees we will call the *initial pruning* of a tree.

Definition 18. The *initial pruning* of a tree x, denoted $Pr_0(x)$, is the union of all infinite linear trees contained by x.

The initial pruning of a tree essentially has all the "finite" parts of it pruned away. If a tree is finite, it's initial pruning is the empty set $\emptyset$.

Consider tree $x = \{\mathbf{10}\} \cup \{\mathbf{0}^i\mathbf{1}^j : i \geq 0, j \geq 0\}^-$,



Thus we have $y = Pr_0(x) = \{\mathbf{0}^i\mathbf{1}^j : i \geq 0, j \geq 0\}^-$ where the string $\mathbf{10}$ was pruned away as it is not included by any infinite linear tree contained by y. Note how $\{\mathbf{0}\}^*$ is in a way more fundamental to y than $\{\mathbf{1}\}^*$ since $\{\mathbf{0}\}^*$ has an infinite number of infinite linear trees contained by y branching off of it. We will call such a linear tree interior since upon removing it from y it will be regenerated upon prefix closure. That is $(y - \{\mathbf{0}\}^*)^- = y$.

Definition 19. A set x is interior to tree y iff $(y - x)^- = y$.

With a concept of interior trees we can define successive rounds of pruning.

Definition 20. Let the pruning of tree x, denoted $Pr(x)$, be the union of all interior infinite linear trees contained by x.

Definition 21. Let the *prunings* of tree x, denoted $\Phi(x)$ be the smallest collection of trees closed under intersection, closed under Pr and containing $Pr_0(x)$.

Note that $\Phi(x)$ exists because it is the intersection of all collections satisfying the above conditions and this intersection must also satisfy the above conditions.

When a tree does not have $\{\mathbf{0}, \mathbf{1}\}^*$ homeomorphically embedded in it it has the nice property that $\Phi(x)$ contains a sequence of prunings ending with $\emptyset$. For trees in $\mathbb{T}_h$ however, the sequence does not terminate with $\emptyset$. For this reason $\Phi(x)$ will be most useful in analyzing trees outside of $\mathbb{T}_h$. The following theorem establishes this result.

Theorem 7. *For non-empty trees x, $x = Pr(x)$ iff there is a homeomorphic embedding from $\{\mathbf{0},\mathbf{1}\}^*$ covering x.*

Proof. If x is not the union of infinite linear trees then x must include some string w which is not in $Pr(x)$ and for which no string in $\{\mathbf{0},\mathbf{1}\}^*$ can be mapped to w in a homeomorphic embedding from $\{\mathbf{0},\mathbf{1}\}^*$ to x. Thus we now consider only x which are the union of infinite linear trees.

Suppose there exists a homeomorphic embedding f from $\{\mathbf{0},\mathbf{1}\}^*$ covering x. Consider any infinite linear tree s contained by x. By Theorem 4, $f^{-1}(s)$ must also be an infinite linear tree. Since $f^{-1}(s)$ is interior to $\{\mathbf{0},\mathbf{1}\}^*$ so must $f(f^{-1}(s))$ and thus s is interior to x. Thus $x = Pr(x)$.

Suppose $x = Pr(x)$. For any w in x, $x|_w$ can not be an infinite linear tree since otherwise $(wx|_w)^-$ is not interior and thus $x \neq Pr(x)$. Thus for every w in x, we may choose a shortest $g(w)$ such that $\{\mathbf{1},\mathbf{0}\} \subseteq x|_{wg(w)}$. Map $f(\varepsilon)$ to $g(\varepsilon)$. For every string w and letter $a \in \{\mathbf{0},\mathbf{1}\}$, map $f(wa)$ to $f(w)ag(f(w)a)$. f is a homeomorphic embedding from $\{\mathbf{0},\mathbf{1}\}^*$ covering x. ☐

The prunings of a tree have a well-ordered structure where $Pr(x)$ acts as the successor operator and a pruning is "less than" another pruning when it contains the other (as a set). Note how usually the subset relationship is treated as a "less than" relationship whereas here we have the superset relationship as "less than".

A typical conclusion from set theory using Zorn's lemma is as follows.

Theorem 8. *Consider any partially ordered set $(U, \leq)$ with the following properties for every subset S of U*

- $\sup(S)$ *exists in U and is unique*
- *if S is total, then for any $x \leq \sup(S)$ there must exists some $y \in S$ such that $x \leq y$*

For any such $(U, \leq)$ it must hold that for any $z \in U$ and function $f : U \mapsto U$ with $x \leq f(x)$ for all $x \in U$ there exists a unique smallest subset M of U containing z and closed under f and sup. *Furthermore such a M must be well-ordered under $\leq$.*

Proof. The outline of the proof is to consider the collection of all subsets S of M such that S is well-ordered under $\leq$, z is in S and for every $x \in S$ and $y \in M$ with $y \leq x$ we have $y \in S$. Zorn's lemma is used to prove the collection must have a maximal element and then it can be shown that the maximal element must equal M. ☐

Corollary 3. *For any tree x, $\Phi(x)$ is well-ordered under the superset relation $\supseteq$.*

Proof. We apply Theorem 8 with trees ordered by the superset relation (not subset). Set intersection qualifies as sup when superset is the "less than" relationship. The function Pr acts as function f and $Pr_0(x)$ as z. $\square$

For any given tree, we will want to know what is the largest pruning whose pruning is $\emptyset$. This will be called the trunk.

Definition 22. The *trunk* of a tree x, denoted $\langle x\rangle$, is the union of all trees y in $\Phi(x)$ such that $Pr(y) = \emptyset$. If the trunk is linear then $\langle x\rangle$ may be treated as a sequence of strings. $\langle x\rangle_n$ is defined to be the string of length n in linear $\langle x\rangle$.

3.3 $\mathbf{\Omega_2}(\mathbb{T})$ well-quasi-orderings

Definition 23. For $a \in \{\mathbf{0}, \mathbf{1}\}$ and any tree x with linear trunk, define the following sequences:

$$x\|_a \quad := \quad \{x|_{\langle x\rangle_{k_i} a} : \langle x\rangle_{k_i}\overline{a} \in \langle x\rangle\}_i$$

Note that these sequences may be infinite, finite and even empty.

Definition 24. Given quasi-ordering A and sequences of trees $\{x_i\}_{i<\alpha}$ and $\{y_i\}_{i<\beta}$ with $\alpha \leq \omega$ and $\beta \leq \omega$,

$$\{x_i\}_{i<\alpha} \quad \leq_A^* \quad \{y_i\}_{i<\beta}$$

iff there exists a one-to-one order preserving mapping $f : \mathbb{N} \mapsto \mathbb{N}$ such that for all $i < \alpha$, we have $x_i \leq_A y_{f(i)}$.

In [7] Higman's Lemma is generalized to all transfinite sequences for which the following lemma is an easy special case:

Lemma 3 (Higman's Extended). *If A is a well-quasi-orderings, then A^* is a well-quasi-ordering.*

Definition 25. For any quasi-ordering A and tree sequence $\{x_i\}_{i<\alpha}$ with $\alpha \leq \omega$, $\{x_i\}$ is said to be *perpetual in A* (or A-perpetual) iff

$$\{x_i\}_i \quad \leq_A^* \quad \{x_{i+1}\}_i$$

Note that with this definition the trivial empty sequence $\emptyset$ is trivially perpetual for any quasi-ordering. The only non-trivial perpetual sequences are infinite.

Some useful facts about A-perpetual sequences are that for any $k > 0$,

$$\{x_i\}_i \quad \leq^*_A \quad \{x_{i+k}\}_i$$

and

$$\{x_k\}_i \quad \leq^*_A \quad \{x_{i+k}\}_i$$

The first term in the last inequality is the constant sequence of x_k and is true because x_k must be less than (in A) some infinite sequence of x_i beyond x_k.

Definition 26. For any tree x and quasi-ordering A, x is said to be *perpetual in A* (or A-perpetual) iff x has a linear trunk and for both $a \in \{\mathbf{0}, \mathbf{1}\}$ $x\|_a$ is A-perpetual and if x is finite then x is $\emptyset$.

Theorem 9 (Perpetual Monotonicity). *For any A in $\mathbf{\Omega_2}(\mathbb{T})$ and A-perpetual trees x and y if for both $a \in \{\mathbf{0}, \mathbf{1}\}$*

$$x\|_a \quad \leq^*_A \quad y\|_a$$

then

$$x \quad \leq_A \quad y$$

Proof. Because the branches form perpetual sequences we can find a homeomorphic embedding h from $\langle x \rangle$ to $\langle y \rangle$ where for all $va \in \langle x \rangle$, $a \in \{\mathbf{0}, \mathbf{1}\}$, we have

$$x|_{v\overline{a}} \quad \leq_A \quad y|_{h(v)\overline{a}}$$

Define

$$t_n \quad := \quad a\, y|_{h(va)} \cup \overline{a}\, x|_{v\overline{a}} \cup \{\varepsilon\}$$

where $va = \langle x \rangle_{n+1}$. Since

$$x \quad = \quad y_{h(\varepsilon)} \mathop{\triangle}_{i=0}^{\infty} [t_i]_{\langle x \rangle_i}$$

showing that this tree series in descending shows by Continuity that $x \leq_A y_{h(\varepsilon)} \leq_A y$, completing the proof.

Since

$$\left. \left(y_{h(\varepsilon)} \mathop{\bigtriangleup}_{i=0}^{n-1} [t_i]_{\langle x \rangle_i} \right) \right|_{\langle x \rangle_n} \quad = \quad y|_{h(\langle x \rangle_n)}$$

we complete the proof by observing that with $va = \langle x \rangle_{n+1}$ we have

$$\begin{array}{rcl} y|_{h(va)} & \leq_A & y|_{h(v)a} \\ x|_{v\overline{a}} & \leq_A & y|_{h(v)\overline{a}} \\ t_n & \leq_A & y|_{h(v)} \end{array}$$

□

Corollary 4. *For any ordering A in $\mathbf{\Omega_2}(\mathbb{T})$ and A-perpetual tree x with linear trunk and $w \in \langle x \rangle$*

$$x \leq_A x|_w$$

Proof. For both $a \in \{\mathbf{0}, \mathbf{1}\}$,

$$\begin{array}{rcl} x|_w\|_a & = & \{x\|_{ai+k_a}\}_i \\ x\|_a & \leq_A^* & \{x\|_{ai+k_a}\}_i \\ & \leq_A^* & x|_w\|_a \end{array}$$

where k_a is the number of letters $\overline{a}$ in w. □

Definition 27. For any ordinal α,

$$\mathbb{C}(\alpha) \quad := \quad \{x \in \overline{\mathbb{T}_h} : \phi(\Phi(x)) \leq \alpha\}$$

Since $\Phi(x)$ always contains a least $\emptyset$, $\mathbb{C}(0)$ is empty. Since $\Phi(x) = \{\emptyset\}$ iff x is finite, $\mathbb{C}(1) = \mathbb{T}_f$.

Definition 28. For any ordinal α and any quasi-ordering A, let $\mathbb{C}(\alpha, A, 0)$ be $\mathbb{C}(\alpha)$ union all A-perpetual trees in $\mathbb{C}(\alpha + 1)$.

Definition 29. For any ordinal α and any quasi-ordering A,

$$\mathbb{C}(\alpha, A, m+1) \quad := \quad \mathbb{C}(\alpha, A, m) \quad \cup \quad \{(x \curlywedge y) : x, y \in \mathbb{C}(\alpha, A, m)\}$$

Theorem 10. *For any ordinal* α *and ordering* A *well-quasi-ordered over* $\mathbb{C}(\alpha)$*,*

$$\mathbb{C}(\alpha+1) \quad = \quad \bigcup_{i \geq 0} \mathbb{C}(\alpha, A, i)$$

Proof. Consider any ordinal α, well-quasi-ordering A and tree x in $\mathbb{C}(\alpha+1)$. Let $b(x)$ denote

$$\{w \in \langle x \rangle : \langle x|_w \rangle \text{ not linear}\}$$

Note that $x|_w \in \mathbb{C}(\alpha+1)$ iff $w \in \langle x \rangle$. We show that $b(x)$ must be finite. If $b(x)$ were infinite then it must contain some infinite linear tree s. Furthermore, for every member $w \in s$ there must exist string v such that wv is in $\langle x \rangle$ but not in s. Therefore s is interior to $\langle x \rangle$ and thus contained by $Pr(\langle x \rangle)$ but this must be $\emptyset$. Thus $b(x)$ must be finite. Thus x can be constructed by a finite number of application of ($\curlywedge$) on sub-branches with linear trunks.

We now show that any such sub-branches with linear trunks can be constructed by a finite number of application of ($\curlywedge$) on an A-perpetual tree thus completing the proof. Consider any $y \in \mathbb{C}(\alpha)$ with linear trunk. If there is no $w \in \langle x \rangle$ such that $y|_w$ is A-perpetual, then for at least one of $a \in \{\mathbf{0}, \mathbf{1}\}$ there is a sequence of $\{w_i\}_i$ such that for no $i < j$ is $y|_{w_i a} \leq_A y|_{w_j a}$ but then A is not well-quasi-ordered over $\mathbb{C}(\alpha)$. ☐

Lemma 4. *For any ordinal* α*, integer* $n \geq 0$*, and quasi-ordering* A *in* $\mathbf{\Omega_2}(\mathbb{T})$*, if* A *is well-quasi-ordered over* $\mathbb{C}(\alpha, A, n)$ *then* A *is well-quasi-ordered over* $\mathbb{C}(\alpha, A, n+1)$*).*

Proof. Consider any infinite sequence $\{x_i\}_i$ over

$$\mathbb{C}(\alpha, A, n+1) \setminus \mathbb{C}(\alpha, A, n)$$

Showing that there must exist a $j < k$ such that $x_{i_j} \leq_A x_{i_k}$ completes the proof.

The sequence $\{x_i|_{\mathbf{0}}\}_i$ must be over $\mathbb{C}(\alpha, A, n)$ and thus there must be an infinite subsequence $\left\{x_{i_j}|_{\mathbf{0}}\right\}_j$ increasing in A.

Similarly the sequence $\left\{x_{i_j}\big|_{\mathbf{1}}\right\}_j$ must also be over $\mathbb{C}(\alpha, A, n)$ and thus there must exist some i_j and i_k such that $j < k$ and $x_{i_j}\big|_{\mathbf{1}} \leq_A x_{i_k}|_{\mathbf{1}}$. By Monotonicity it must hold that $x_{i_j} \leq_A x_{i_k}$. ☐

Lemma 5. *For any ordinal* α *and quasi-ordering* A *in* $\mathbf{\Omega_2}(\mathbb{T})$*, if* A *is well-quasi-ordered over* $\mathbb{C}(\alpha)$ *then* A *is well-quasi-ordered over* $\mathbb{C}(\alpha, A, 0)$*).*

Proof. If $\alpha = 0$, $\mathbb{C}(\alpha, A, 0))$ is $\{\emptyset\}$ and thus well-quasi-ordered.

Consider $\alpha > 0$ and any infinite sequence $\{x_i\}_i$ over $\mathbb{C}(\alpha, A, 0) \setminus \mathbb{C}(\alpha)$.

The sequence

$$\{x_i\|_{\mathbf{1}}\}_i$$

must be over sequences over $\mathbb{C}(\alpha)$ thus by Lemma 3 (Higman's Lemma Extension) there must be an infinite subsequence

$$\left\{x_{i_j}\|_{\mathbf{1}}\right\}_j$$

that is increasing in $\leq^*_A$.

Similarly the sequence

$$\left\{x_{i_j}\|_{\mathbf{0}}\right\}_j$$

must also be over sequences over $\mathbb{C}(\alpha)$ thus by Lemma 3 (Higman's Lemma Extension) there must exist some i_j and i_k such that $j < k$ and

$$x_{i_j}\|_{\mathbf{0}} \quad \leq^*_A \quad x_{i_j}\|_{\mathbf{0}}$$

By Theorem 9 (Perpetual Monotonicity) it must hold that $x_{i_j} \leq_A x_{i_k}$.

The existence of a $x_{i_j} \leq_A x_{i_k}$ must also hold for any sequence over $\mathbb{C}(\alpha, A, 0)$ and thus A is well-quasi-ordered. □

Theorem 11. *Every ordering in* $\mathbf{\Omega_2}(\mathbb{T})$ *is well-quasi-ordered.*

Proof. Consider any ordering A in $\mathbf{\Omega_2}(\mathbb{T})$. By Lemma 5, finite induction with Lemma 4 and Theorem 10 we conclude that if $\mathbb{C}(\alpha)$ is well-quasi-ordered then $\mathbb{C}(\alpha + 1)$ is also well-quasi-ordered. We apply trans-finite induction to conclude that A is well-quasi-ordered over $\overline{\mathbb{T}_h}$.

A must be well-quasi-ordered over all trees since all trees in $\mathbb{T}_h$ are order-equivalent and greater than every tree in $\overline{\mathbb{T}_h}$ by Corollary 2 and Corollary 1. □

3.4 $\mathbf{\Omega_{2L}}(\mathbb{T})$ well-orderings

In this section we prove three basic theorems and a collection of lemmas which together will prove that all members of $\mathbf{\Omega_{2L}}(\mathbb{T})$ are well-ordered.

Theorem 12 (Perpetual Totality). *If A is total then $\leq^*_A$ is total over A-perpetual sequences.*

Proof. Consider any A-perpetual sequences $\{x_i\}_i$ and $\{y_i\}_i$. If there exists some k such that for all $j \geq k$ we have $x_j \leq_A y_k$ then

$$\begin{aligned} \{x_i\}_i &\leq_A^* \{x_{k+i}\}_i \\ &\leq_A^* \{y_k\}_i \\ &\leq_A^* \{y_i\}_i \end{aligned}$$

If there does not exist k such that for all $j \geq k$ we have $x_j \leq_A y_k$ then there is a one-to-one order-preserving mapping f such that for all i we have $y_i \leq_A x_{f(i)}$ and thus $\{y_i\}_i \leq_A^* \{x_i\}_i$ □

For any tree x, it will be useful to write $\{x\}^*$ to denote the infinite sequence

$$(x, x, x, ...)$$

Theorem 13 (Infinite Lexicographic Collapse). *For any A in $\mathbf{\Omega_{2L}}(\mathbb{T})$, tree y and tree sequence $\{x_i\}$, if $x_i <_A y$ for all $i \geq 0$ then*

$$\left(\bigcup_{i=0}^{\infty} \mathbf{0}^i \mathbf{1} x_i \right)^- \quad \leq_A \quad (y \curlywedge \emptyset)$$

Proof. For all $n \geq 0$ define

$$t_n \quad := \quad (y \curlywedge \emptyset) \mathop{\triangle}_{i=0}^{n-1} [(x_i \curlywedge (y \curlywedge \emptyset))]_{\mathbf{0}^i}$$

which results in

$$t_n \quad = \quad (\mathbf{0}^n (y \curlywedge \emptyset))^- \cup \bigcup_{i=0}^{n-1} \mathbf{0}^i \mathbf{1} x_i$$

For all n we have

$$\begin{aligned} x_n &<_A y \\ (x_n \curlywedge (y \curlywedge \emptyset)) &\leq_A (y \curlywedge \emptyset) \quad \text{by Lexicography} \\ &\leq_A t_n|_{\mathbf{0}^n} \end{aligned}$$

thus by Continuity

$$\left(\bigcup_{i=0}^{\infty} \mathbf{0}^i \mathbf{1} x_i \right)^- \quad \leq_A \quad (y \curlywedge \emptyset)$$

since

$$\left(\bigcup_{i=0}^{\infty} \mathbf{0}^i \mathbf{1} x_i\right)^- \;=\; \lim_{n\to\infty} t_n \;=\; (y \curlywedge \emptyset) \bigtriangleup_{i=0}^{\infty} [(x_i \curlywedge (y \curlywedge \emptyset))]_{\mathbf{0}^i}$$

$\square$

Theorem 14 (Perpetual Lexicography). *For any* A *in* $\mathbf{\Omega_{2L}}(\mathbb{T})$ *and* A*-perpetual trees* x *and* y *with the trunk of* y *not equal to* $\mathbf{0}^*$*,*

$$\text{if} \quad x\|_{\mathbf{0}} \leq_A^* y\|_{\mathbf{0}} \quad \text{and} \quad x\|_{\mathbf{1}} <_A^* \{y\}^* \quad \text{then} \quad x \leq_A y$$

Proof. There must exist a mapping f from $\langle x\rangle$ to $\langle y\rangle$ such that the following two conditions both hold:

1. For every $v\mathbf{0} \in \langle x\rangle$, $f(v\mathbf{0}) = f(v)$ and $f(v)\mathbf{1} \in \langle y\rangle$
2. For every $v\mathbf{1} \in \langle x\rangle$, $f(v)\mathbf{1} \preceq f(v\mathbf{1})$ and $x|_{v\mathbf{0}} \leq_A y|_{f(v)\mathbf{0}}$

Define

$$t_n \quad := \quad \begin{cases} (x|_{v\mathbf{1}} \curlywedge y|_{f(v)}) & \text{if } v\mathbf{0} = \langle x\rangle_{n+1} \\ (y|_{f(v\mathbf{1})} \curlywedge x|_{v\mathbf{0}}) & \text{if } v\mathbf{1} = \langle x\rangle_{n+1} \end{cases}$$

Since

$$x \quad = \quad y|_{f(\varepsilon)} \bigtriangleup_{i=0}^{\infty} [t_i]_{\langle x\rangle_i}$$

showing that this tree series in descending shows by Continuity that $x \leq_A y$, completing the proof.

Define

$$z_n \quad := \quad y|_{f(\varepsilon)} \bigtriangleup_{i=0}^{n-1} [t_i]_{\langle x\rangle_i}$$

We want to show that

$$t_n \quad \leq_A \quad z_n|_{\langle x\rangle_n}$$

for all $n \geq 0$.

For $v\mathbf{1} = \langle x\rangle_{n+1}$ we have

$$\begin{aligned} x|_{v\mathbf{0}} &\leq_A y|_{f(v)\mathbf{0}} \\ y|_{f(v\mathbf{1})} &\leq_A y|_{f(v)\mathbf{1}} \\ t_n &\leq_A y|_{f(v)} \\ &\leq_A z_n|_v \end{aligned}$$

For $v\mathbf{0} = \langle x \rangle_{n+1}$ we have

$$\begin{array}{rcl} f(v)\mathbf{1} & \in & \langle y \rangle \\ y & \simeq & y|_{f(v)\mathbf{1}} \\ x|_{v\mathbf{1}} & <_A & y \\ & <_A & y|_{f(v)\mathbf{1}} \\ t_n & \leq_A & y|_{f(v)} \\ & \leq_A & z_n|_v \end{array}$$

by Corollary 4 and Lexicography.

□

Corollary 5. *For any A in $\mathbf{\Omega_2}(\mathbb{T})$, tree x and A-perpetual tree y with linear trunk,*

$$\exists\, w\mathbf{1} \in \langle y \rangle \ \left[\ x \leq_A y|_{w\mathbf{0}}\ \right] \ \Rightarrow \ (y \curlywedge x) \simeq_A y$$

Proof. By Growth and Corollary 4 since $(y \curlywedge x)$ must be A-perpetual. □

Now that we have proved the three basic theorems above, we proceed to proving four lemmas which when used with trans-finite induction will prove that $\mathbf{\Omega_{2L}}(\mathbb{T})$ consists of well-orderings.

The later two lemmas, 8 and 9, are summarized as

1. if $\mathbb{C}(\alpha)$ is total then so is $\mathbb{C}(\alpha, A, 0)$
2. if $\mathbb{C}(\alpha, A, 0)$ is total then so is $\mathbb{C}(\alpha + 1)$

These two lemmas together will form the last step in proving that $\mathbf{\Omega_{2L}}(\mathbb{T})$ is of well-orderings.

The first two lemmas, 6 and 7, are used to prove Lemma 8.

It will be useful to call a set of trees *comparable* to another set of trees if every member of one set is comparable to every other member of the other set.

Lemma 6. *For any ordering A in $\mathbf{\Omega_{2L}}(\mathbb{T})$ and ordinals $\beta \leq \alpha$ with $\mathbb{C}(\alpha)$ total, if $\mathbb{C}(\beta)$ is comparable to $\mathbb{C}(\alpha, A, 0)$ then so is $\mathbb{C}(\beta, A, 0)$*

Proof. Assume the givens with $\mathbb{C}(\beta)$ comparable to $\mathbb{C}(\alpha, A, 0)$. We first show which pairs of subtrees are comparable.

Consider any tree x in $\mathbb{C}(\alpha, A, 0)$ and tree y in $\mathbb{C}(\beta, A, 0)$.

If $\beta = \alpha$ then $x\|_{\mathbf{1}}$ is over $\mathbb{C}(\beta)$ and y is in $\mathbb{C}(\alpha, A, 0)$. If $\beta < \alpha$ then $x\|_{\mathbf{1}}$ is over and y is in $\mathbb{C}(\alpha)$. In either case, every member of $x\|_{\mathbf{1}}$ is comparable to y. Every member of $y\|_{\mathbf{1}}$ is in $\mathbb{C}(\beta)$ and thus comparable to x.

Both $x\|_{\mathbf{1}}$ and $y\|_{\mathbf{1}}$ must be over $\mathbb{C}(\alpha)$, thus by Theorem 12 (Perpetual Totality) both are comparable in $\leq^*_A$. Likewise for $x\|_{\mathbf{0}}$ and $y\|_{\mathbf{0}}$.

Now we complete the proof with three cases.

Consider any pair of trees x and y, one from $\mathbb{C}(\alpha, A, 0)$ and the other from $\mathbb{C}(\beta, A, 0)$, not necessarily in that order.

Consider the first case where both x and y have trunks equal to $\mathbf{0}^*$. By Theorem 9 (Perpetual Monotonicity) x and y must be comparable.

Consider the second case where both x and y do not have trunks equal to $\mathbf{0}^*$. If $x\|_{\mathbf{1}} \geq^*_A \{y\}^*$ then $x \geq_A y$. If $y\|_{\mathbf{1}} \geq^*_A \{x\}^*$ then $y \geq_A x$. Otherwise $x\|_{\mathbf{1}} <^*_A \{y\}^*$ and $y\|_{\mathbf{1}} <^*_A \{x\}^*$. Since either $x\|_{\mathbf{0}} \leq^*_A y\|_{\mathbf{0}}$ or $y\|_{\mathbf{0}} \leq^*_A x\|_{\mathbf{0}}$, by Theorem 14 (Perpetual Lexicography) either $x \leq_A y$ or $y \leq_A x$.

Consider the last case where one tree has a trunk equal to $\mathbf{0}^*$ and the other does not. With out loss of generality, assume x has a $\mathbf{0}^*$ trunk. If there exists some z in $x\|_{\mathbf{1}}$ such that $y \leq_A z$ then $y \leq_A x$ by Growth. Otherwise there must be some $\mathbf{0}^i\mathbf{1}$ in $\langle y \rangle$ such that for every z in $x\|_{\mathbf{1}}$ we must have

$$\begin{aligned} z &<_A y \\ &<_A y|_{\mathbf{0}^i\mathbf{1}} \end{aligned}$$

and thus by Theorem 13 (Infinite Lexicographic Collapse) we have

$$\begin{aligned} x &\leq_A (y|_{\mathbf{0}^i\mathbf{1}} \curlywedge \emptyset) \\ &\leq_A y|_{\mathbf{0}^i} \\ &\leq_A y \end{aligned}$$

□

Lemma 7. *For any ordering A in $\mathbf{\Omega_{2L}}(\mathbb{T})$ and ordinals $\beta \leq \alpha$ with $\mathbb{C}(\alpha)$ total, if $\mathbb{C}(\beta, A, n)$ is comparable to $\mathbb{C}(\alpha, A, 0)$ then so is $\mathbb{C}(\beta, A, n+1)$*

Proof. Assume givens and consider any tree x in $\mathbb{C}(\alpha, A, 0)$ and tree $y \in \mathbb{C}(\beta, A, n+1)$.

If $\mathbb{C}(\beta, A, n)$ is comparable to $\mathbb{C}(\alpha, A, 0)$ then either $x \leq_A y|_0$, in which case $x \leq_A y$, or $y|_0 <_A x$. Likewise either $x \leq_A y|_1$, in which case $x \leq_A y$, or $y|_1 <_A x$.

If x does not have trunk $\mathbf{0}^*$, we have $x \simeq_A (x \curlywedge \emptyset)$ and thus

$$y = (y|_1 \curlywedge y|_0)$$

$$\begin{aligned} &\leq_A (y|_1 \curlywedge x)\\ &\leq_A (y|_1 \curlywedge (x \curlywedge \emptyset))\\ &\leq_A (x \curlywedge \emptyset)\\ &\leq_A x \end{aligned}$$

Otherwise x does have trunk $\mathbf{0}^*$. If there exists an z in $x\|_\mathbf{1}$ such that $y|_1 \leq_A z$ then

$$\begin{aligned} y &= (y|_1 \curlywedge y|_0)\\ &\leq_A (y|_1 \curlywedge x)\\ &\leq_A (z \curlywedge x)\\ &\leq_A x \end{aligned}$$

Otherwise, since $\mathbb{C}(\alpha)$ is total, $z <_A y|_1$ for every z in $x\|_\mathbf{1}$ and thus by Theorem 13 (Infinite Lexicographic Collapse) we have

$$\begin{aligned} x &\leq_A (y|_1 \curlywedge \emptyset)\\ &\leq_A y \end{aligned}$$

□

Now with the previous two lemmas we can prove the following lemma.

Lemma 8. *For any ordering A in $\mathbf{\Omega_{2L}}(\mathbb{T})$ and ordinal α, if $\mathbb{C}(\alpha)$ is total then $\mathbb{C}(\alpha, A, 0)$ is also total.*

Proof. For any $\beta \leq \alpha$ with $\mathbb{C}(\alpha)$ total, we can apply finite induction on Lemma 7 and use Theorem 10 to conclude that if $\mathbb{C}(\beta, A, 0)$ is comparable to $\mathbb{C}(\alpha, A, 0)$ then $\mathbb{C}(\beta + 1)$ is also comparable to $\mathbb{C}(\alpha, A, 0)$. Furthermore, by Lemma 6, $\mathbb{C}(\beta + 1, A, 0)$ must also be comparable to $\mathbb{C}(\alpha, A, 0)$.

With this result and the base case of $\mathbb{C}(0, A, 0) = \{\emptyset\}$ being comparable to $\mathbb{C}(\alpha, A, 0)$, we can apply trans-finite induction to conclude that $\mathbb{C}(\alpha, A, 0)$ is total. □

We now look at the last lemma we need towards proving that all orderings in $\mathbf{\Omega_{2L}}(\mathbb{T})$ are well-orderings.

Lemma 9. *For any quasi-ordering A in $\mathbf{\Omega_{2L}}(\mathbb{T})$ and any ordinal α, if A is total over $\mathbb{C}(\alpha, A, 0)$ then A is total over $\mathbb{C}(\alpha + 1)$*

Proof. Let R_n denote

$$\bigcup_{i=0}^{n} \mathbb{C}\left(\alpha, A, i\right) \times \mathbb{C}\left(\alpha, A, n-i\right)$$

We proceed with induction on n with the following inductive proposition:

$$\text{For all } (x, y) \in R_n \qquad x \leq_A y \quad \text{or} \quad y \leq_A x$$

For $n = 0$, x and y are comparable in A by hypothesis.

Assume case $n = N$ holds. Consider any $(x, y) \in R_{N+1}$. Since $(x|_1, y|_1) \in R_N$ we must have three cases relating $x|_1$ and $y|_1$:

Case $y|_1 \simeq_A x|_1$. Either $x|_0 \leq_A y|_0$ or $y|_0 \leq_A x|_0$ and thus either $x \leq_A y$ or $y \leq_A x$.

Case $y|_1 <_A x|_1$: Since $(x, y|_0) \in R_N$ either $y|_0 \leq_A x$ in which case $y \leq_A x$ from Lexicography or $x \leq_A y|_0$ in which case $x \leq_A y$ from Growth.

Case $y|_1 >_A x|_1$: Likewise.

By finite induction, A is total over $\mathbb{C}\left(\alpha, A, n\right)$ for any n and thus over $\mathbb{C}\left(\alpha + 1\right)$ by Theorem 10. □

Theorem 15. *Every ordering in* $\mathbf{\Omega_{2L}}(\mathbb{T})$ *is well-ordered.*

Proof. By Theorem 11, $\mathbf{\Omega_{2L}}(\mathbb{T})$ is well-quasi-ordered. Showing all members of $\mathbf{\Omega_{2L}}(\mathbb{T})$ are total completes the proof.

Consider any ordering A in $\mathbf{\Omega_{2L}}(\mathbb{T})$. Showing that A is total over $\overline{\mathbb{T}_h}$ completes the proof since all trees in $\mathbb{T}_h$ are equivalent (Corollary 2) and greater than every tree in $\overline{\mathbb{T}_h}$.

By Lemma 8 and Lemma 9, if $\mathbb{C}\left(\alpha\right)$ is total then $\mathbb{C}\left(\alpha + 1\right)$ is also total. By trans-finite induction we must have A total over $\overline{\mathbb{T}_h}$. □

As with finite trees, the set of ordering for infinite tree has a leanest ordering, as defined in [1]. With the result of $\mathbf{\Omega_{2L}}(\mathbb{T})$ consisting of total well-quasi-orderings, the example proof of existence for finite trees in [1] can be re-applied to infinite trees.

3.5 Order Type

In this section we find the order-type of the leanest ordering of $\mathbf{\Omega_{2L}}(\mathbb{T})$. The first step is finding the initial segment of M which has the order-type of the first uncountable ordinal.

Definition 30.

$$\begin{aligned}\mathbb{D}_0 &:= \{\emptyset\}\\ \mathbb{D}_{\alpha+1} &:= \mathbb{D}_\alpha \cup \{x : \forall n(x|_{\mathbf{0}^n\mathbf{1}} \in \mathbb{D}_\alpha)\}\\ \mathbb{D}_{\sup \alpha_i} &:= \bigcup_{\alpha_i} \mathbb{D}_{\alpha_i}\end{aligned}$$

Theorem 16.

$$\bigcup_{\alpha\in\omega_1} \mathbb{D}_\alpha = \mathbb{T}_a$$

where $\mathbb{T}_a$ *is the set of all trees for which any contained linear subtree has only a finite number of letters* $\mathbf{1}$*.*

Proof. By transfinite induction over the well-orderedness of tree prunings, it can be shown that any tree in $\mathbb{T}_a$ must also be in some $\mathbb{D}_\alpha$. Conversely, any tree with a linear subtree containing an infinite number of ones must not be in any $\mathbb{D}_\alpha$ because that would imply that there exists an infinite sequence of ordinals which is strictly descending. □

Definition 31. Over the set $\mathbb{T}_a$ define:

$$f(x) = \inf\{\alpha : x \in \mathbb{D}_\alpha\}$$

and let F be the quasi-ordering such that

$$x \leq_F y$$

if and only if $f(x) \leq f(y)$.

Theorem 17. $F \in \mathbf{\Omega_{2L}}(\mathbb{T}_a)$

Proof. Showing that F satisfies the Growth, Monotonicity, Lexicography and the Nil principles is fairly straightforward. We now consider the Continuity principle. Given any decreasing extended tree series in $\mathbf{\Omega_{2L}}(\mathbb{T}_a)$:

$$y \mathop{\triangle}_{i=0}^{\infty} [x_i]_{s_i}$$

we must have some N for which all $k > N$ there in no symbol $\mathbf{1}$ in s_i. This implies that for all $k > N$

$$f\left(y \mathop{\triangle}_{i=0}^{\infty} [x_i]_{s_i}\right) \leq f\left(y \mathop{\triangle}_{i=0}^{k} [x_i]_{s_i}\right)$$

By repeated finite monotonicity we must have the limit tree series less than or equal to y in F. □

Lemma 10. *For any trees x and y in $\mathbb{T}_a$, if $f(x) < f(y)$ then $x <_M y$, where M is the leanest ordering of $\mathbf{\Omega_{2L}}(\mathbb{T})$.*

Proof. We proceed with transfinite induction over ω_1, the first uncountable ordinal. For any given $\alpha < \omega_1$ our inductive hypothesis is that $f(x) < f(y) = \alpha$ implies $x <_M y$.

Consider any $f(x) < f(y) = \alpha$ and suppose the inductive hypothesis holds for all ordinals strictly less than α. For some integer n we must have

$$x|_{\mathbf{0}^m\mathbf{1}} \quad <_M \quad y|_{\mathbf{0}^n\mathbf{1}}$$

for all integers m. Thus by Infinite Lexicographic Collapse we have

$$\begin{aligned} x \quad &\leq_M \quad (y|_{\mathbf{0}^n\mathbf{1}} \curlywedge \emptyset) \\ &\leq_M \quad y \end{aligned}$$

We must also have $x <_M y$ since otherwise there would exist an ordering strictly leaner than M which is equal to M for all tree comparisons within $\{x : f(x) < \alpha\}$ and equal to F for all other tree comparisons. □

Theorem 18. *The order type of the leanest ordering over $\mathbb{T}_a$ is ω_1, the first uncountable ordinal.*

Proof. Since the leanest ordering over $\mathbb{T}_a$ is a subset of F, M has an order type equal to or greater than the order type of F. The order type of F must be ω_1 since if F has a countable order type, a tree in $\mathbb{T}_a$ can be constructed to required F to have a higher order type. Conversely, F can have no order type greater than ω_1 since for any tree in $\mathbb{T}_a$ there will be at most a countable number of trees in a lower order-position in F. The order type of M can also not be greater than ω_1 since the order-type of M within an F-equivalence-class will always be countable. □

We now expand our analysis to determine the order type of the leanest ordering over all infinite trees. We can start by expanding the definition of $\mathbb{D}$.

Definition 32.

$$\begin{aligned} \mathbb{D}_0(x) \quad &:= \quad \{x\} \\ \mathbb{D}_{\alpha+1}(x) \quad &:= \quad \mathbb{D}_\alpha(x) \cup \{x : \forall n (x|_{\mathbf{0}^n\mathbf{1}} \in \mathbb{D}_\alpha(x))\} \\ \mathbb{D}_{\sup \alpha_i}(x) \quad &:= \quad \bigcup_{\alpha_i} \mathbb{D}_{\alpha_i}(x) \end{aligned}$$

Theorem 19.

$$\mathbb{T} \setminus \mathbb{T}_h \quad = \bigcup_{\substack{\alpha \in \omega_1 \\ x \in \mathbb{T}_b}} \mathbb{D}_\alpha(x)$$

where $\mathbb{T}_b$ is the set $\{\emptyset\}$ union the set of all trees with an infinite linear trunk containing an infinite number of **1** *symbols.*

Proof. Every tree not in $\mathbb{T}_h$ has a well-ordered set of prunings with $\emptyset$ as the first pruning. Via trans-finite induction it can be shown that any tree outside $\mathbb{T}_h$ must be constructible from $\mathbb{D}_\alpha(x)$ and trees with trunks containing an infinite number of **1** symbols. □

Definition 33. For any cardinality ξ, let Ψ denote the normal function which maps any ordinal α to the first ordinal whose set of predecessors:

- contains all $\Psi_\xi(\beta)$ for which $\beta < \alpha$
- contains the first *strict* upper bound to any subset of cardinality ξ

If ξ is finite, then Ψ_ξ is $\alpha \mapsto \omega \cdot \alpha$. For $\xi = \aleph_0$ we have $\Psi_{\aleph_0}(i) = \omega_1 \cdot i$ for $i < \omega$, however $\Psi_{\aleph_0}(\omega)$ is much larger than $\omega_1 \cdot \omega$.

Theorem 20. *The leanest ordering of $\mathbf{\Omega_{2L}}(\mathbb{T})$ has the order-type of the successor of the first fixed point of $\Psi_{\aleph_0}$.*

Proof. Let M denote the leanest ordering of $\mathbf{\Omega_{2L}}(\mathbb{T})$. The set $\mathbb{T} \setminus \mathbb{T}_h$ is partitioned by the collection of sets

$$\bigcup_{\alpha<\omega_1} \mathbb{D}_\alpha(x)$$

where x is $\emptyset$ or any tree with an M-perpetual linear trunk containing an infinite number of **1** symbols. Furthermore, these partitions form a well-ordering. The mapping from the order-type of the well-ordering of partitions to the order-type in M is the mapping $\Phi_{\aleph_0}$. As ordered by M, a partition must contain strict upper bounds of subsets of cardinality $\aleph_0$, exactly as $\Phi_{\aleph_0}$ enumerates ordinals.

This means M over $\mathbb{T} \setminus \mathbb{T}_h$ must have the order-type of the first fixed point of $\Phi_{\aleph_0}$. With $\mathbb{T}_h$ as an equivalence in M greater than all other trees we must have the order-type of M as the successor. □

Conjecture 1. *The leanest ordering of $\mathbf{\Omega_{2L}}(\mathbb{T})$ has an order type greater than or equal to the order type of any well-ordering in $\mathbf{\Omega_2}(\mathbb{T})$.*

3.6 Regular Trees

Stepping from finite trees to infinite trees resulted in a very large step in order type. The set of all infinite trees with only a finite number of distinct subtrees is an intermediate step from finite trees to infinite trees. This section discusses some of the properties of regular trees.

All of the previous results about infinite trees apply to regular trees, with the exception of order-type. The leanest ordering over regular trees only has order-type $\epsilon_{\epsilon_{\epsilon_{\dots}}}$, the first critical epsilon number.

When restricted to regular trees, perpetual trees are much simpler. For either side of the trunk of a regular perpetual tree there must be a maximum branch (in comparison to all of the other branches on the same side). Any such tree must be equivalent to the perpetual tree with only the maximum branches on the given sides.

For the set $\mathbb{T}_a$ restricted to the set of regular trees, $\mathbb{T}_r$, this greatly reduces the order-type so much so that $\mathbb{T}_a \cup \mathbb{T}_r$ only has the order-type of finite trees, ϵ_0. Any perpetual tree in $\mathbb{T}_a \cup \mathbb{T}_r$ is equivalent to a simple single left branch perpetual tree which, by Theorem 13, is merely the predecessor of $(x \Yup \emptyset)$ where x is the successor of the single left branch. Adding predecessors will not increase the order-type of $\mathbb{T}_a \cup \mathbb{T}_r$ from the order-type of the finite trees, thus the order-type remains at ϵ_0.

References

[1] Nachum Dershowitz and E. Castedo Ellerman. Leanest quasi-orderings. In J. Giesl, editor, *Proceedings of the Sixteenth International Conference on Rewriting Techniques and Applications (Nara, Japan)*, volume 3467 of *Lecture Notes in Computer Science*, pages 32–45, Berlin, April 2005. Springer-Verlag.

[2] H. Friedman. *The metamathematics of the Higman and Kruskal theorems.* Unpublished, 19??

[3] G. Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society (3)*, 2(7):326–336, September 1952.

[4] J. Ketonen and R. Solovay. Rapidly growing Ramsey functions. *Ann. Math.*, 113:267–314, 1981.

[5] L. Kirby and J. Paris. Accessible independence results for Peano arithmetic. *Bulletin London Mathematical Society*, 14:285–293, 1982.

[6] Laver. Better-quasi orderings and a class of trees. *Studies in foundations and combinatorics, advances in mathematics supplementary studies*, 1:31–48, 1978.

[7] C. S. J. A. Nash-Williams. On well-quasi-ordering transfinite sequences. *Proceedings of the Cambridge Philosophical Society*, 61(1):33–39, 1965.

[8] M. Okada and A. Steele. Ordering structures and the Knuth-Bendix completion algorithm. In *Proceedings of the Allerton Conference on Communication, Control, and Computing, Monticello, IL*, 1988.

[9] E. M. Reingold and X. Shen. More nearly optimal algorithms for unbounded searching, part II: The transfinite case. *SIAM J. Comput.*, 20(1):184–208, February 1991.